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Quotient Stacks and String Orbifolds

Eric Sharpe
Department of Physics
Box 90305
Duke University
Durham, NC 27708
ersharpe@cgtp.duke.edu

In this short review we outline some recent developments in understanding string orbifolds. In particular, we outline the recent observation that string orbifolds do not precisely describe string propagation on quotient spaces, but rather are *literally* sigma models on objects called quotient stacks, which are closely related to (but not quite the same as) quotient spaces. We show how this is an immediate consequence of definitions, and also how this explains a number of features of string orbifolds, from the fact that the CFT is well-behaved to orbifold Euler characteristics. Put another way, many features of string orbifolds previously considered “stringy” are now understood as coming from the target-space geometry; one merely needs to identify the correct target-space geometry.

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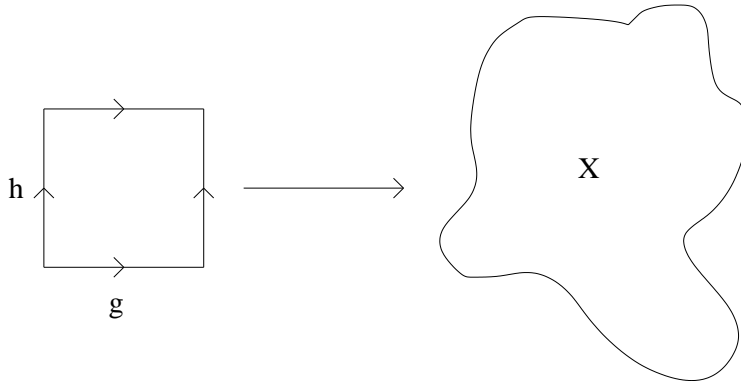


Figure 1: Contribution to the (g, h) twisted sector of a string orbifold on T^2

1 Introduction

One often hears that string orbifolds [1, 2, 3] describe strings propagating on quotient spaces (with some ‘stringy’ effects at singularities). Of course, a string orbifold is not described in terms of maps into a quotient space, but rather is set up in terms of group actions on covering spaces. For example, the partition function of a string orbifold is of the form

$$Z \propto \sum Z(\text{twisted sector})$$

e.g., the partition function of a string orbifold on T^2 can be written

$$Z_{T^2} = \frac{1}{|G|} \sum_{g,h, gh=hg} Z_{g,h}$$

where each $Z_{g,h}$ is a sigma model from a square into X , such that the sides of the square are related by the action of $g, h \in \Gamma$, as illustrated in figure 1.

We shall argue later that, in fact, the twisted sector sum and the functional integral within each twisted sector of a string orbifold partition function are both duplicated by a sum over maps into something known as a quotient *stack*, an object which is closely related to the quotient space. Quotient stacks look mostly like quotient spaces, but have “extra information” over any singularities of the quotient space, which makes quotient stacks much better behaved than quotient spaces. The fact that a sum over maps into a quotient stack duplicates the sums in a string orbifold is a smoking gun for an interpretation of string orbifolds as sigma models on quotient stacks. We shall also outline how one can define a classical action for a sigma model on a stack, which reproduces string orbifolds in the special case of quotient stacks.

Of course, this is a rather strong conclusion to most physicists. Mathematicians, on the other hand, may find this result considerably more natural. After all, one way of thinking

about quotient stacks is as extraordinarily overcomplicated ways of describing group actions on covers, which is the language used in string orbifolds. Also, quotient stacks have the structure of a ‘generalized space’ – many stacks can be considered to be slight generalizations of ordinary spaces, just as noncommutative geometry gives a way of generalizing ordinary geometry. In particular, one often hears¹ that one can do differential geometry on stacks, a necessary (but by no means sufficient) condition to be able to compactify a string.

Although we shall describe classical actions for strings on stacks, generalizing string orbifold actions, and describe how a sum over maps into a quotient stack duplicates both the twisted sector and functional integral sums in a string orbifold, we should emphasize that a tremendous amount of work remains to be done to check whether the notion of string compactification on stacks is indeed sensible. We have only set up the basics in [7].

One question one might well ask is simply, why bother? If at the end of the day, we are merely using quotient stacks as a radically overcomplicated way of describing group actions on covers, then there is hardly a point. Part of the reason for our interest in string orbifolds specifically is that this approach seems to give a new geometric understanding of certain physical features of string orbifolds. Another reason is that this has nontrivial implications for the role string orbifolds play in the rest of string theory – for example, constructions of moduli spaces of string vacua may need to be slightly rethought, considering that deformation theory of a string orbifold would have to be understood in terms of deformation theory of a quotient stack, instead of a quotient space.

The work outlined herein was originally done in an attempt to understand whether it is possible to compactify a string on a stack (here thought of as a slight generalization of a space). One sometimes hears that, for example, one can perform differential geometry on stacks, which would be one necessary condition; however, a tremendous amount of work must be done to check whether this notion is sensible. We shall outline the beginnings of a program to understand string compactification on stacks – specifically, we shall describe sigma models on stacks. Now, such a proposal is meaningless without interesting examples, string orbifolds appear to offer the first nontrivial examples.

It should also be said that there exists a group of mathematicians who already use quotient stacks to describe string orbifolds. Indeed, as described above, quotient stacks are an overcomplicated way to describe group actions on covering spaces, and also possess extra structure that gives them an interpretation as a sort of ‘generalized space,’ so that, for example, one can make sense of differential geometry on a quotient stack. However, as far as the author has been able to determine, the mathematicians in question have not done any of the work required to justify the claim that a string orbifold CFT coincides with the CFT for a string compactified on a quotient stack, or even to justify the claim that the notion of string compactification on a stack makes sense. They do not seem to have attempted to

¹Readable references describing, in detail, differential geometry on stacks are hard to find. For this reason, reference [7] included a lengthy discussion of differential geometry on stacks.

study sigma models on stacks, and do not even realize why this is relevant. They also do not seem to be aware of even the most elementary physical implications of such a claim, such as the fact that, to be consistent, any deformations of a string orbifold CFT would have to be interpreted in terms of deformations of the quotient stack, rather than the quotient space. Thus, mathematicians reading this paper should interpret our work as the beginnings of a program to fill in the logical steps that they seem to have omitted. In fact, this paper was written for a physics audience; mathematicians are encouraged to instead read lecture notes [8] we shall publish shortly.

2 String orbifolds do not live on X/Γ

During the preparation of [7], we had numerous conversations with physicists claiming that string orbifolds describe strings on quotient spaces. As a result, we feel compelled to spend some time examining this more closely. Of course, no one would claim that a string orbifold is the same thing as a sigma model on a quotient space, if for no other reason than the fact that one sums over maps into the cover, instead of the quotient. However, some might argue that this is merely an artifact of the description, and to counter such arguments, we shall spend a bit of time examining the issue more closely. In the process, we shall also gain some perspective that will be useful later.

Given any space X , we can define a category of maps into X . Specifically,

1. Objects in this category are continuous maps $f : Y \rightarrow X$ from any other topological space Y into X .
2. Morphisms $(Y_1 \xrightarrow{f_1} X) \longrightarrow (Y_2 \xrightarrow{f_2} X)$ are continuous maps $\lambda : Y_1 \rightarrow Y_2$ such that the following diagram commutes:

$$\begin{array}{ccc} Y_1 & \xrightarrow{\lambda} & Y_2 \\ f_1 \downarrow & & \downarrow f_2 \\ X & = & X \end{array}$$

It is a standard result that a topological space X is determined by this category. In other words, if we know all of the maps into a space, we can reconstruct the space.

Let us consider string orbifolds. Can string orbifolds be literally interpreted as sigma models on quotient spaces? Not unless the orbifold group Γ acts freely.

The most efficient way to study this problem is to first describe the twisted sector maps more elegantly. Instead of talking about maps from twisted sectors into X , as in figure 1,

an equivalent and more elegant description is as a pair

$$\left(E \xrightarrow{\pi} \Sigma, E \xrightarrow{f} X \right)$$

where

Σ is the string worldsheet,
 $E \rightarrow \Sigma$ is a principal Γ -bundle, and
 $f : E \rightarrow X$ is a continuous Γ -equivariant map.

In this description, a twisted sector is the same thing as an equivalence class of bundles. To see how to recover the original description, simply restrict E to a maximally-large contractible open subset of Σ . (If we describe the Riemann surface Σ in terms of a polygon with sides identified, such an open set would be the interior of the polygon.) Over a contractible open set, E is trivializable, so pick some section s . The ordinary description of maps from twisted sectors into X is precisely the action of f on that section s , and the group elements acting at the boundary determine to what extent s fails to be a global section. Precisely because $f : E \rightarrow X$ is Γ -equivariant, knowing the action of f on such a section s completely determines f , i.e., there is no extra information contained in the map $f : E \rightarrow X$ that is not also contained in the twisted sector map into X .

Given such a pair $(E \xrightarrow{\pi} \Sigma, E \xrightarrow{f} X)$, we can derive a continuous map $g : \Sigma \rightarrow X/\Gamma$ into the quotient space. Specifically, for any point $y \in \Sigma$, let e be any point in the fiber of E over y . If we denote the canonical projection $X \rightarrow X/\Gamma$ by π_0 , then define $g : \Sigma \rightarrow X/\Gamma$ by, $g(y) = (\pi_0 \circ f)(e)$. It is straightforward to check that this is well-defined and continuous.

To what extent is a pair $(E \xrightarrow{\pi} \Sigma, E \xrightarrow{f} X)$ the same thing as a map $\Sigma \rightarrow X/\Gamma$? Given the former, we can construct the latter; however, in a well-defined sense, if Γ does not act freely, then there are more maps of the former form than of the latter. In other words, pairs $(E \xrightarrow{\pi} \Sigma, E \xrightarrow{f} X)$ seem to be defining maps into a space related to X/Γ but containing “extra information” over any singularities.

In the next section, we shall argue that a string orbifold, although not quite a sigma model on a quotient space X/Γ , appears to be literally a sigma model on something called a quotient *stack* $[X/\Gamma]$. (We shall argue this first by noting that a sum over maps into a “quotient stack” duplicates both the sum over twisted sectors and the functional integral within each twisted sector, a smoking gun for a sigma model interpretation, and then propose a classical action for a sigma model on a stack.)

What is a quotient stack? To be very brief, a quotient stack is an example of what is sometimes known as a “generalized space.” For example, instead of possessing a set of points, it has a *category* of points. In general, such spaces can be defined in terms of the category

of maps into them. This is somewhat analogous to noncommutative geometry, where spaces are defined by the algebra of functions on them. Here, instead of working with the algebra of functions on the space, one works with the category of continuous maps into the space. This may sound somewhat cumbersome, but it is actually an ideal setup for string sigma models.

In passing, we should also mention that the idea of defining spaces in terms of the maps into them is a commonly-used setup in algebraic geometry (see discussions of “Grothendieck’s functor of points” in, for example, [4, section II.6] or [5, section VI]).

How are quotient stacks related to quotient spaces? For example, when the orbifold group Γ acts freely, the quotient stack $[X/\Gamma]$ and the quotient space X/Γ are homeomorphic. So, one way of thinking about quotient stacks is that they look like quotient spaces, except that they have some “extra structure” over the singularities. That “extra structure” makes quotient stacks much better behaved than quotient spaces, and is responsible for features of string orbifolds than people have labelled “stringy” in the past.

3 Unravel definitions – path integral sums

Quotient stacks $[X/\Gamma]$ are defined by the category of maps from all topological spaces into $[X/\Gamma]$. In particular, a continuous map from any topological space Σ into $[X/\Gamma]$ is a pair

$$\left(E \xrightarrow{\pi} \Sigma, E \xrightarrow{f} X \right)$$

where

$E \rightarrow \Sigma$ is a principal Γ -bundle, and
 $f : E \rightarrow X$ is a continuous Γ -equivariant map.

But, we argued earlier that these are precisely the things one literally sums over in a string orbifold – by summing over maps into a quotient stack, one duplicates both the twisted sector sum as well as the sum over maps within each twisted sector. So, in other words, a string orbifold is *literally* a sum over maps from the worldsheet into the quotient stack $[X/\Gamma]$, a ‘smoking gun’ for an interpretation as a sigma model. Put another way, after unraveling definitions, string orbifolds appear to be sigma models on quotient stacks, at least judging by the path integral sum. In the next section, we shall describe a proposal for a classical action for a sigma model on a stack, and further justify this conclusion.

Some readers might find this conclusion to be somewhat extreme – after all, physicists have not explicitly considered stacks in the past. Indeed, part of the point of [7] was to set up the basics required for physicists to begin to consider string compactification on stacks.

One needs some nontrivial examples of such compactifications to begin to take such a notion seriously, and string orbifolds appear to offer such an example.

4 Classical actions for sigma models on stacks

Now, just because a string orbifold is literally a sum over maps into $[X/\Gamma]$ does not itself imply that a string orbifold is necessarily a sigma model on $[X/\Gamma]$ – for that to be the case, we also need the action associated to each map to be the same, so that the path integral is the same *weighted* sum over maps. In this section we shall describe a natural proposal for a classical action for a sigma model on a stack, and check that this not only duplicates standard sigma models when the target stack is an honest space, but also duplicates string orbifolds (down to the $|G|^{-1}$ factors in partition functions, when the target is a global quotient stack).

Let \mathcal{F} be a stack, with atlas X . (We shall only attempt to describe sigma models on stacks with atlases.) For readers not well-acquainted with stacks, for X to be an atlas for \mathcal{F} implies that

- implicitly there is also a fixed map $X \rightarrow \mathcal{F}$ (which is required to be a surjective local homeomorphism)
- for any space Y and map $Y \rightarrow \mathcal{F}$, the fibered product $Y \times_{\mathcal{F}} X$ is an honest space, not a stack.

For example, if \mathcal{F} is a space, not just a stack (spaces are special cases of stacks), then \mathcal{F} is its own atlas, and $Y \times_{\mathcal{F}} X = Y$ for any Y . For another example, suppose \mathcal{F} is a quotient stack $[X/G]$, with G discrete and acting by diffeomorphisms on a smooth space X . In such a case, X is an atlas for $[X/G]$. In this case, $Y \times_{[X/G]} X$ is a principal G -bundle over Y , partially specifying the map $Y \rightarrow [X/G]$, and the projection map $Y \times_{[X/G]} X \rightarrow X$ is the G -equivariant map from the total space of the bundle to X , specifying the rest of the map $Y \rightarrow [X/G]$.

Now, the natural description of a sigma model with target \mathcal{F} , formulated on (base) space Y , is a sum over equivalence classes² of maps $Y \rightarrow \mathcal{F}$, weighted by $\exp(iS)$, where the classical action S is formulated as follows. Fix a map $\phi : Y \rightarrow \mathcal{F}$. If we let³ $\Phi : Y \times_{\mathcal{F}} X \rightarrow X$ denote the second projection map (implicitly encoding part of the map $\phi : Y \rightarrow \mathcal{F}$), then the natural proposal for the bosonic part of the classical action for a sigma model on \mathcal{F} is

²A sigma model path integral is a sum over maps, after all, hence one must take equivalence classes in order to make sense out of such a sum.

³Note that since both $Y \times_{\mathcal{F}} X$ and X are ordinary spaces, Φ is a map in the ordinary sense of the term.

given by [7]

$$\int d^2\sigma (\pi_1^* \phi^* G_{\mu\nu}) \pi_1^* h^{\alpha\beta} \left(\frac{\partial \Phi^\mu}{\partial \sigma^\alpha} \right) \left(\frac{\partial \Phi^\nu}{\partial \sigma^\beta} \right) \quad (1)$$

where $h^{\alpha\beta}$ is the worldsheet metric, $\phi^* G$ denotes the pullback of the metric on \mathcal{F} to Y (metrics on \mathcal{F} are described in terms of their pullbacks), $\pi_1 : Y \times_{\mathcal{F}} X \rightarrow Y$ is the projection map, and this action is integrated over a lift⁴ of Y to $Y \times_{\mathcal{F}} X$.

A few examples should help clarify this description:

1. Suppose \mathcal{F} is an ordinary space. Then the path integral is a sum over maps into that space, and as $Y \times_{\mathcal{F}} X = Y$ (taking the atlas X to be \mathcal{F} itself), we see that $\Phi = \phi$, and so in this case the classical action proposed above duplicates the usual classical action,

$$S \sim \int d^2\sigma (\phi^* G_{\mu\nu}) h^{\alpha\beta} \frac{\partial \phi^\mu}{\partial \sigma^\alpha} \frac{\partial \phi^\nu}{\partial \sigma^\beta} + \dots$$

as well as the path integral sum. Thus, the description above duplicates sigma models on ordinary spaces.

2. Suppose $\mathcal{F} = [X/G]$, where X is smooth and G is a nontrivial action of a discrete group by diffeomorphisms. Then the path integral is a sum over equivalence classes of maps $Y \rightarrow [X/G]$, which is to say, equivalence classes of principal G -bundles on Y together with G -equivariant maps from the total space of the bundle into X . It is easy to check that the proposed classical action above duplicates the usual classical action for a string orbifold. Also, by summing over (equivalence classes of) maps $Y \rightarrow \mathcal{F}$, note we are summing over both twisted sectors as well as maps within any given twisted sector.

Now, for each such map $\phi : Y \rightarrow \mathcal{F}$, there are $|G|$ lifts of Y to the total space of the bundle (which is $Y \times_{[X/G]} X$), *i.e.*, $|G|$ twisted sector maps, as they usually appear in physics.

Note we are only summing over equivalence classes of bundles, not all possible twisted sector maps. However, we can trivially sum over all possible twisted sector maps, at the cost of overcounting by $|G|$. Hence, we can equivalently describe this in terms of a sum over twisted sector maps, but weighted by $|G|^{-1}$. Hence we recover both the path integral sum and the overall multiplicative factor of $|G|^{-1}$ appearing in string orbifold partition functions, for example, the one-loop partition function

$$Z(T^2) = \frac{1}{|G|} \sum_{\substack{g, h \in G \\ gh=hg}} Z_{(g,h)}$$

⁴Sensible essentially because the (projection) map $\pi_1 : Y \times_{\mathcal{F}} X \rightarrow Y$ is a surjective local homeomorphism.

Thus, we see that the natural definition of a sigma model on a stack duplicates not only sigma models on ordinary spaces, but also string orbifolds when the target is a quotient stack, even down to the $|G|^{-1}$ factor appearing in partition functions.

We should take this opportunity to also note that this description of sigma models does not make any assumptions concerning the dimension of the base space Y – classically there are analogues of ‘string’ orbifolds in every dimension, all obtained precisely by gauging the action of a discrete group on the target space of a sigma model.

So far we have only recovered known results; let us now try something new. Suppose the target \mathcal{F} is a gerbe. For simplicity, we shall assume that \mathcal{F} is the canonical trivial G -gerbe on a space X . Such a gerbe is described by the quotient stack $[X/G]$, where the action of G on X is trivial. Using the notion of sigma model on a stack as above, one quickly finds that the path integral for this target space is the same as the path integral for a sigma model on X , up to an overall multiplicative factor (equal to the number of equivalence classes of principal G -bundles on Y). As overall factors are irrelevant in path integrals, the result appears to be that a string on the canonical trivial gerbe is the same as a string on the underlying space. More generally, it is natural to conjecture that strings on flat gerbes are equivalent to strings on underlying spaces, but with flat B fields. In particular, such a result would nicely dovetail with the well-known fact that a coherent sheaf on a flat gerbe is equivalent to a ‘twisted’ sheaf on the underlying space, the same twisting that occurs in the presence of a B field. (For physicists, this is an alternative to the description in terms of modules over Azumaya algebras that has recently been popularized [17].)

So far we have only discussed classical actions for sigma models on stacks, but there is much more that must be done before one can verify that the notion of a sigma model on a stack is necessarily sensible. In effect, we have only considered local behavior, but in order to be sure this notion is sensible after quantization, one also needs to consider global phenomena. Such considerations were the source of much hand-wringing when nonlinear sigma models on ordinary spaces were first introduced (see for example [18]), and must be repeated for stacks.

5 More results

So far we have argued that a string orbifold is literally a sigma model on a quotient *stack*, as opposed to a quotient space. What does this do for us? We just saw that this naturally explained the structure of twisted sectors; next we shall outline how this also gives geometric perspectives on many other features of string orbifolds, previously considered “stringy.”

Specifically:

1. Smoothness. String orbifold CFT's do not suffer from any singularities; they behave as if they were sigma models on smooth spaces. This led to the old lore that “strings smooth out singularities.” However, quotient stacks $[X/\Gamma]$ are always smooth (for X smooth and Γ acting by diffeomorphisms), and so one naturally expects that a sigma model on $[X/\Gamma]$ should always be well-behaved. The old lore “strings smooth out singularities” is merely a consequence of misunderstanding the target space geometry; nothing “stringy” is really involved.
2. B fields. Another often-quoted fact concerning string orbifolds is that the B field should somehow have nonzero holonomy about shrunken exceptional divisors [11, 12, 13, 14, 15]. Understanding string orbifolds as sigma models on quotient stacks gives a natural understanding of what is meant by such claims. Specifically, the “extra information” contained by a quotient stack over singularities of the quotient space is a gerbe⁵, that precisely duplicates standard results on B fields at quotient singularities. A gerbe is, after all, just a special kind of stack, so the reader should not be surprised to find information about B fields given in the geometry of stacks.
3. The role of equivariance. Bundles and sheaves on $[X/\Gamma]$ are the same thing as Γ -equivariant bundles and sheaves on X . So, we now gain a new perspective on the role that equivariance plays in describing fields on string orbifolds.
4. Twist fields. Ordinarily the low-energy spectrum of a string compactification is determined by the cohomology of the target space. However, we argue in [7] that for stacks matters are slightly more interesting, and the low-energy spectrum is determined by the cohomology of an auxiliary stack, known as the associated inertia group stack $I_{[X/\Gamma]}$. (This result appears very obscure to most physicists, many of whom have traditionally expected an understanding of twist fields in terms of a cohomology of the quotient *space*; mathematicians familiar with stacks, on the other hand, will recognize this result.) This stack differs from $[X/\Gamma]$ precisely when the quotient stack cannot be understood as an ordinary space. More to the point, $I_{[X/\Gamma]}$ naturally encodes the “twists” that give twist fields their name, and in fact has the form [10]

$$I_{[X/\Gamma]} \cong \coprod_{[g]} [X^g/C(g)]$$

which of course compares well with orbifold Euler characteristics [9]

$$\chi_{orb}(X, \Gamma) = \sum_{[g]} \chi(X^g/C(g))$$

where $[g]$ denotes conjugacy classes of Γ and $C(g)$ the centralizer of $g \in \Gamma$.

In [7] we work through the points above in much greater detail, and also describe how some other features of string orbifolds are clarified.

⁵A gerbe is a formal structure corresponding to B fields, just as bundles correspond to gauge fields.

So far, we have argued that understanding string orbifolds as sigma models on quotient stacks gives a natural geometric explanation to many features of string orbifolds that were previously considered “inherently stringy.”

But what new things can we do with quotient stacks? We shall list a few general directions:

1. Deformation theory. In constructing moduli spaces of string vacua, physicists have assumed string orbifolds describe strings on quotient spaces. If string orbifolds actually describe strings on quotient stacks, then these old arguments must be reconsidered – for example, a blowup modulus must be interpreted as some hypothetical Kähler modulus of the quotient stack, and not in terms of a (partial) resolution of the quotient space. We have given indirect evidence in [7] that such considerations might ultimately be equivalent to working with a resolution with a nonzero B field on the exceptional divisor, but a tremendous amount of work remains to be done to check whether this is actually the case.
2. New string compactifications. There are more stacks than just global quotient stacks. So, given our preliminary work on string compactification on stacks (bolstered by the nontrivial example of string orbifolds), one could begin seriously studying compactifications on other stacks.
3. M-theory⁶ orbifolds. If one were careful, in the past one could have objected that orbifolds in M-theory could hardly be considered well-understood. After all, orbifolds in string theory possess twisted sectors, twist fields, and many other features which naively seemed to be “inherently stringy” – it was not at all clear how one could make sense of such things in M-theory. Now, however, we can understand these matters better. We can define an M-theory orbifold to be, M-theory compactified on a quotient stack. Then, for example, membranes on a quotient stack have an obvious twisted-sector-type structure. One might be able to do quite a bit more with this description – for example, it may be possible to directly understand the Horava-Witten [16] E_8 multiplets as arising naturally, whereas in their original description these multiplets had to be added in manually.

6 Conclusions

In this short note we have outlined some recent results on string orbifolds. Specifically, we have outlined how string orbifolds seem to be literally sigma models on quotient stacks (not

⁶We use “M-theory” in the sense of, quantum theory underlying eleven-dimensional supergravity, as opposed to some hypothetical master theory.

quotient spaces). We have also described how, from this new perspective, many properties of string orbifolds that were formerly considered “stringy,” actually seem to have a simple geometric understanding in terms of the target space geometry.

However, a tremendous amount of work remains to be done, not only to better understand what it means to compactify a string on a stack (indeed, to check whether this is indeed a sensible notion), but also to understand the consequences of this perspective (such as deformation theory of string orbifolds). In the conclusions of [7] we have given a lengthy list of further topics that could be pursued.

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